

Toward an uncountable analogue of Gallai's Theorem for colorings of the plane

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Abstract

In this paper we prove that if S is any finite configuration of points in \mathbb{Z}^2 , then any finite coloring of \mathbb{E}^2 must contain uncountably many monochromatic subsets homothetic to S . We extend a result of Brown, Dunfield, and Perry on 2-colorings of \mathbb{E}^2 to any *finite* coloring of \mathbb{E}^2 .

1 Introduction

Let \mathbb{E}^2 denote the Euclidean Plane. Many authors have considered the question, for which finite sets S in \mathbb{E}^2 is it true that if the points of \mathbb{E}^2 are colored in finitely many colors, there must be some monochromatic subset congruent to S ? (For an extensive treatment of this and related problems, see [7].) Call such a set S *Ramsey (for \mathbb{E}^2)*. In [6], it is proved that all right triangles are Ramsey for two colors. No equilateral triangles are Ramsey, for we can avoid a monochromatic equilateral triangle with side length d by coloring the plane with vertical strips of width $\sqrt{3}d/2$, alternating in red and blue. In [4], it is conjectured that if a 2-coloring of \mathbb{E}^2 contains no monochromatic equilateral triangles with unit side length, then it contains monochromatic equilateral triangles of *all* other sizes.

We might also consider replacing the word *congruent* above with *homothetic* (where A is *homothetic* to B if A can be mapped onto B by a translation and a dilation). Gallai's Theorem solves this problem in the affirmative—see below.

Finally, let us further expand our consideration to include all monochromatic subsets *similar* to S (where A is *similar* to B if A can be mapped onto B by some sequence of translations, dilations, rotations, and reflections). In Theorem 4 of [3], the authors show that in any

given 2-coloring of \mathbb{E}^2 there exist uncountably many $r \in \mathbb{R}^+$ so that there is a monochromatic equilateral triangle with side length r ; hence there exist uncountably many monochromatic sets similar to any given equilateral triangle. It is this result that we strengthen in this paper, especially in Theorem 8 below.

Throughout, we let $[k] = \{1, \dots, k\}$.

2 Main Results

We begin with a definition.

Definition 1. A *rectangle* in \mathbb{E}^2 is a set of four points

$$\{(x, y), (x + d_1, y), (x, y + d_2), (x + d_1, y + d_2)\},$$

with $d_1, d_2 > 0$. A *rectangle* in \mathbb{Z}^2 is a set of four points

$$\{(x, y), (x + d_1, y), (x, y + d_2), (x + d_1, y + d_2)\},$$

with $x, y, d_1, d_2 \in \mathbb{Z}$ and $d_1, d_2 > 0$. A *square* in \mathbb{E}^2 (respectively, \mathbb{Z}^2) is a rectangle in \mathbb{E}^2 (respectively, \mathbb{Z}^2) with $d_1 = d_2$.

Note that in this paper, all rectangles and squares have sides parallel to the axes. The proof of the following lemma is left to the reader.

Lemma 2. Let $\mathbb{Z} \times \mathbb{Z}$ be colored in finitely many colors; then there exists some monochromatic rectangle.

Throughout this paper, we will rely heavily on factoring \mathbb{E}^2 into cosets, as in $\mathbb{E}^2 / \mathbb{Z} \times \mathbb{Z}$. Each coset $C \in \mathbb{E}^2 / \mathbb{Z} \times \mathbb{Z}$ is of the form

$$(x, y) + (\mathbb{Z} \times \mathbb{Z}), \quad (x, y) \in [0, 1) \times [0, 1),$$

and so C is an infinite grid; hence Lemma 2 applies to C (even though C is not equal as a set to $\mathbb{Z} \times \mathbb{Z}$). This yields the following result:

Theorem 3. For each $r \in \mathbb{R}^+$ and for every finite coloring of \mathbb{E}^2 there exist uncountably many monochromatic rectangles with side lengths that are integer multiples of r .

Proof. Let $r\mathbb{Z}$ denote $\{rn : n \in \mathbb{Z}\}$. Then $\mathbb{E}^2 / r\mathbb{Z} \times r\mathbb{Z}$ is a collection of cosets of the form

$$(x, y) + (r\mathbb{Z} \times r\mathbb{Z}), \quad (x, y) \in [0, r) \times [0, r).$$

A fixed coset C is a grid that is a translation of $r\mathbb{Z} \times r\mathbb{Z}$. By Lemma 2, C contains a monochromatic rectangle. \square

“Gallai’s Theorem”, which first appeared in the literature in [5], refers to one of two results:

Theorem 4 (Gallai’s Theorem on \mathbb{Z}^2). Let S be a finite subset of \mathbb{Z}^2 . Then any finite coloring of \mathbb{Z}^2 contains a monochromatic subset homothetic to S .

Theorem 5 (Gallai’s Theorem on \mathbb{E}^2). Let S be any finite subset of \mathbb{E}^2 . Then any finite coloring of \mathbb{E}^2 contains a monochromatic subset homothetic to S .

(For a discussion of Gallai’s Theorem as well as a proof, see [7], p. 508.) From Theorem 4, we see that any finite coloring of $\mathbb{Z} \times \mathbb{Z}$, hence any of our coset “grids”, will contain a monochromatic square. Using this result, we can prove a variation on Theorem 3 in which all of the monochromatic rectangles are similar to one another. Let the *aspect ratio* of a rectangle denote the ratio of a rectangle’s width to its height.

Theorem 6. For each $r \in \mathbb{R}^+$ and for every finite coloring of \mathbb{E}^2 , there exist uncountably many rectangles with aspect ratio r .

Proof. Consider $\mathbb{E}^2 / r\mathbb{Z} \times \mathbb{Z}$. Any coset C is a grid, and hence by Theorem 4 contains a “square” of the form $\{(rn, m), (r(n+d), m), (rn, m+d), (r(n+d), m+d)\}$, which corresponds to a rectangle with width rd and height d . \square

In [1], the authors try to find the smallest n so that any 2-coloring of $[n] \times [n]$ contains a monochromatic square; they show $n \geq 13$. In [2], the authors prove that $n = 15$ with the aid of computers. Hence we may give the following improvement of Theorem 6 for two colors:

Theorem 7. Let $0 < r < 1$. For all 2-colorings of the unit square there exist uncountably many monochromatic rectangles with aspect ratio r .

Proof. Let the unit square be 2-colored, and let $0 < r < 1$. We need only consider the subset $[0, r] \times [0, 1]$ of the unit square. Let $A = \{0, \frac{1}{15}, \frac{2}{15}, \dots, \frac{14}{15}\}$, and let $rA = \{0, \frac{r}{15}, \frac{2r}{15}, \dots, \frac{14r}{15}\}$. Consider the collection of cosets

$$[0, r] \times [0, 1] / rA \times A$$

Each coset is a 15-by-15 grid, hence by [2] contains a monochromatic “square”, which is a rectangle with aspect ratio r . \square

Now let us consider equilateral triangles. In [3], the following appears as Theorem 4:

For every two[-]coloring of \mathbb{E}^2 , there exist an uncountable number of values of r , where $r \in \mathbb{R}^+$, such that an equilateral triangle of side r exists monochromatically.

We extend this result to a stronger version that applies to any *finite* coloring of \mathbb{E}^2 . Let \mathcal{T} be the *unit equilateral triangle* $\{(0,0), (1,0), (1/2, \sqrt{3}/2)\}$.

Theorem 8. For any finite coloring of \mathbb{E}^2 there exist uncountably many $r \in \mathbb{R}^+$ such that there exist uncountably many monochromatic equilateral triangles with side length r homothetic to \mathcal{T} . Furthermore, for any 2-coloring of the unit square there exist uncountably many $r \in (0,1]$ such that there exist uncountably many monochromatic equilateral triangles of side length r homothetic to \mathcal{T} .

Proof. For the proofs of both claims, we will consider two copies of \mathbb{E}^2 , which we will think of as vector spaces—the first copy (the domain) with respect to the basis $\langle(1,0), (0,1)\rangle$, and the second copy (the range) with respect to the basis $\langle(r,0), (\frac{r}{2}, \frac{r\sqrt{3}}{2})\rangle$, with r to be chosen later. We will also need the vector-space isomorphism $\varphi : \mathbb{E}^2 \rightarrow \mathbb{E}^2$ that maps

$$\begin{aligned} (1,0) &\mapsto (r,0) \\ \text{and} \\ (0,1) &\mapsto \left(\frac{r}{2}, \frac{r\sqrt{3}}{2}\right). \end{aligned}$$

Notice that φ sends a *square* to a *rhombus*.

To prove the first claim, fix $r \in \mathbb{R}^+$ and suppose the range \mathbb{E}^2 is finitely colored by $\chi : \mathbb{E}^2 \rightarrow [k]$. Let χ induce a coloring χ' of the domain \mathbb{E}^2 via φ^{-1} , i.e. let $\chi'(x,y) = \chi(\varphi(x,y))$. Each coset $C \in \mathbb{E}^2 / \mathbb{Z} \times \mathbb{Z}$ contains a monochromatic square by Theorem 4. The image of such a square under φ is a rhombus with side length rk for some $k \in \mathbb{Z}^+$, and acute interior angles of measure 60 degrees. This gives a monochromatic equilateral triangle with side length an integer multiple of r homothetic to \mathcal{T} . Letting r range over \mathbb{R}^+ gives the desired result.

To prove the second claim, let $0 < r \leq \frac{2}{45}$ and let the unit square $[0,1] \times [0,1]$ be 2-colored by $\chi : [0,1] \times [0,1] \rightarrow [2]$. Now $\varphi([0,15] \times$

$[0, 15]$) is a subset of the unit square; let χ induce a 2-coloring χ' of $[0, 15] \times [0, 15]$ via φ^{-1} , where $\chi'(x, y) = \chi(\varphi(x, y))$. Let $A = \{0, 1, 2, \dots, 14\}$. By [2], any coset in $[0, 15] \times [0, 15] / A \times A$, which is a 15-by-15 grid, contains a monochromatic square. As before, the image under φ of such a square contains a monochromatic equilateral triangle with side lengths (small) integer multiples of r homothetic to \mathcal{T} . Now letting r range over uncountably many values in $(0, \frac{2}{45}]$ that are linearly independent over \mathbb{Z} gives the desired result. \square

Let us generalize and summarize what we have done so far:

Theorem 9. Let S be a finite configuration of points in the integer lattice $\mathbb{Z} \times \mathbb{Z}$. In any finite coloring of the plane \mathbb{E}^2 , there exist uncountably many monochromatic homothetic copies of S .

Proof. Consider the collection of cosets $\mathbb{E}^2 / \mathbb{Z} \times \mathbb{Z}$. A fixed coset C is a translation of $\mathbb{Z} \times \mathbb{Z}$. By Gallai's theorem, each coset contains a monochromatic subset homothetic to S . \square

3 Conclusions

The results given here rely on partitions of the plane into “nice” cosets, or cosets skewed by a linear transformation. The proof of Theorem 8 could be adapted to any rhombus or parallelogram, hence any triangle.

Problem 1. Show that for any finite coloring of \mathbb{E}^2 and any 4-point configuration in the plane, some color class must contain uncountably many homothetic copies of the configuration.

It seems natural to conjecture that any finite configuration in the plane—which must appear in some color class by Gallai's Theorem on \mathbb{E}^2 —must appear in fact uncountably many times.

Problem 2. Show that for any finite coloring of \mathbb{E}^2 and any finite $S \subset \mathbb{E}^2$, some color class must contain uncountably many homothetic copies of S .

A solution to either of these problems would presumably require a partition of the plane more clever than the ones given here.

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